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A Survey of E-Test Spaces

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Test spaces (or quasimanuals) were introduced by D. Foulis and C. Randall in 1972 as a language for describing the empirical sciences. They emphasized that test spaces give a direct description of laboratory operations. Information is lost in moving from a test space to its corresponding logic, and for this reason, test spaces provide a more fundamental description of a physical system. Recently, A. Dvurečenskij and S. Pulmannová have introduced a generalization of test spaces, called D-test spaces, in order to include a description of unsharp measurements. In the present work, we consider an equivalent framework called an effect test space or E-test space for short. This gives an alternative viewpoint that is believed to be simpler and more natural. Moreover, the proofs of results are more direct. This framework also has connections with some of the recent work of D. Foulis, R. Greechie, M. K. Bennett, and G. Rüttimann.

1. INTRODUCTION

This article presents a survey of recent work on E-test spaces. The concept of an E-test space generalizes the test spaces (or quasimanuals) of Foulis and Randall (1972; Randall and Foulis, 1973) in order to include a description of unsharp measurements. Moreover, E-test spaces are equivalent to the D-test spaces of Dvurečenskij and Pulmannová (1994) and unify the recent work of various investigators (Foulis et al., 1993, n.d.).

Morphisms, bimorphisms, and tensor products of E-test spaces and the relationships to their counterparts for effect algebras are investigated. The universal group of an E-test space is defined and studied. It is stated that any E-test space possesses a universal group that is unique to within an isomorphism. Moreover, the universal group is a test group (Foulis *et al.*, n.d.) that determines the E-test space to within an isomorphism. We finally

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consider compatibility and observables on E-test spaces. Proofs of results cannot be included in this short survey and will appear in a later article.

To motivate our work, a test t is an ideal, perfectly accurate (sharp) measurement. Let x be a possible outcome of an experiment that is relevant to (testable by) t. If x occurs, then t registers yes and if x does not occur, t registers no. For example, a particle detector t tests whether a particle is in a certain region R. For $x \in R$, if a particle is at x, then t clicks and for $x \notin R$, if a particle is at x, then t does not click.

Now make N runs of an experiment, where N is a large integer. For simplicity, suppose that there are only a finite number of outcomes $S(t) = \{x_1, \ldots, x_n\}$ that are tested by t. Let $t(x_i)$ be the number of times x_i occurs, $i = 1, \ldots, n$. Then t is described by the function $t: S(t) \to \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. An effect f is considered to be a submeasurement of t that may not be sharp. Suppose the set of outcomes tested by f is $S(f) = \{x_1, \ldots, x_m\}$, $m \leq n$. Now rerun the experiment N times and let $f(x_i)$ be the number of times that f registers a yes when x_i occurs. If f is unsharp, it may not register yes when $x_i \in S(f)$ occurs. Moreover, on rare occasions, f may register a false yes when $x_i \in S(f)$ does not occur, but then f will give a short count for some other $x_j \in S(f)$. On the average we would have $f(x_i) \leq t(x_i)$ for $i = 1, \ldots, m$. The *confidence function* for f is defined to be $\hat{f}: S(f) \to [0, 1]$, where $\hat{f}(x_i) = f(x_i)/t(x_i)$. This function gives the confidence that f will register yes when x_i occurs, $i = 1, \ldots, m$. We would then have that $\hat{t} = I_{S(t)}$, where $I_{S(t)}$ is the indicator (characteristic) function of the set $S(t)$ and f is sharp if and only if $\hat{f} = I_{S(t)}$; that is, $f(x_i) = t(x_i)$ for all $x_i \in S(f)$.

We now give precise mathematical definitions that are motivated by our previous discussion. Let X be a nonempty outcome space corresponding to a physical system and let $\mathcal{T} \subseteq \mathbb{N}_0^X$. We call $(X, \mathcal{T})$ an *E-test space* if the following conditions hold.

(1) For any $x \in X$ there exists a $t \in \mathcal{T}$ such that $t(x) \neq 0$.
(2) If $s, t \in \mathcal{T}$ with $s \leq t$ [i.e., $s(x) \leq t(x)$ for all $x \in X$], then $s = t$.

Condition (1) states that every outcome is testable by at least one test $t \in \mathcal{T}$ and condition (2) states that tests are not redundant.

We call $f \in \mathbb{N}_0^X$ an *effect* if $f \leq t$ for some $t \in \mathcal{T}$ and denote the set of effects by $\mathcal{E} = \mathcal{E}(X, \mathcal{T})$. The *null effect* is the function $f_0 \in \mathcal{E}$ that satisfies $f_0(x) = 0$ for all $x \in X$. We say that $f, g \in \mathcal{E}$ are:

(i) *orthogonal* ($f \perp g$) if $f + g \in \mathcal{E}$
(ii) *local complements of each other* (f loc g) if $f + g \in \mathcal{T}$
(iii) *perspective* ($f \approx g$) if they share a local complement.

We say that $(X, \mathcal{T})$ is *algebraic* if for $f, g, h \in \mathcal{E}$, $f \approx g$ and $h \perp f$ imply $h \perp g$. If $(X, \mathcal{T})$ is algebraic, then $\approx$ is an equivalence relation and we denote the equivalence class containing $f \in \mathcal{E}$ by $\pi(f)$. Then the logic

$$\Pi(X) = \mathscr{E}/\approx = \{\pi(f): f \in \mathscr{E}\}$$

can be organized into an effect algebra (Foulis and Bennett, 1994) in a natural way. Conversely, any effect algebra is isomorphic to the logic of an algebraic E-test space.

2. MORPHISMS AND BIMORPHISMS

Let X, Y be nonempty sets and let $\phi: X \to Y$. For $f \in \mathbb{N}_0^X$, we define $\hat{\phi}(f): Y \to \mathbb{N}_0 \cup \{\infty\}$ by

$$\hat{\phi}(f)(y) = \Sigma\{f(x): \phi(x) = y\}$$

If $(X, \mathscr{T})$, $(Y, \mathscr{S})$ are E-test spaces, then $\phi: X \to Y$ is a *morphism* if $\hat{\phi}(t) \in \mathscr{S}$ for every $t \in \mathscr{T}$. If ϕ is a bijective morphism and ϕ^{-1} is a morphism, then ϕ is an *isomorphism.* A mapping $\psi: \mathscr{E}(X) \to \mathscr{E}(Y)$ is an *effect morphism* if $\psi(t) \in \mathscr{S}$ for every $t \in \mathscr{T}$ and $f \perp g$ implies that

$$\psi(f + g) = \psi(f) + \psi(g)$$

If ψ is a bijective effect morphism and ψ^{-1} is an effect morphism, then ψ is an *effect isomorphism.* The next result summarizes some of the important properties of morphisms and effect morphisms.

Theorem 2.1. (a) If $\phi: X \to Y$ is a morphism, then $\hat{\phi}: \mathscr{E}(X) \to \mathscr{E}(Y)$ is an effect morphism. (b) A mapping $\phi: X \to Y$ is an isomorphism if and only if $\hat{\phi}: \mathscr{E}(X) \to \mathscr{E}(Y)$ is an effect isomorphism. (c) A mapping $\psi: \mathscr{E}(X) \to \mathscr{E}(Y)$ is an effect isomorphism if and only if there exists an isomorphism $\phi: X \to Y$ such that $\psi = \hat{\phi}$.

The following result gives the connection between effect morphisms on algebraic E-test spaces and effect algebra morphisms (Foulis and Bennett, 1994).

Theorem 2.2. Let $(X, \mathscr{T})$, $(Y, \mathscr{S})$ be algebraic E-test spaces and let $\psi: \mathscr{E}(X) \to \mathscr{E}(Y)$ be an effect morphism. Then $\tilde{\psi}: \Pi(X) \to \Pi(Y)$ given by $\tilde{\psi}(\pi(f)) = \pi(\psi(f))$ is an effect algebra morphism. If ψ is an effect isomorphism, then $\tilde{\psi}$ is an effect algebra isomorphism.

For $f \in \mathbb{N}_0^X$, $g \in \mathbb{N}_0^Y$, we define $f \times g \in \mathbb{N}_0^{X \times Y}$ by $(f \times g)(x, y) = f(x)g(y)$. Let $(X, \mathscr{T})$, $(Y, \mathscr{S})$, $(Z, \mathscr{U})$ be E-test spaces and let $\beta: X \times Y \to Z$. For $f \in \mathscr{E}(X)$, $g \in \mathscr{E}(Y)$, we define $\hat{\beta}(f, g): X \times Y \to \mathbb{N}_0 \cup \{\infty\}$ by $\hat{\beta}(f, g) = \hat{\beta}(f \times g)$. We call β a *bimorphism* if $\hat{\beta}(t, s) \in \mathscr{U}$ for all $t \in \mathscr{T}$, $s \in \mathscr{S}$. A mapping $\alpha: \mathscr{E}(X) \times \mathscr{E}(Y) \to \mathscr{E}(Z)$ is an *effect bimorphism* if it satisfies the following conditions:

(i) $\alpha(t, s) \in \mathscr{U}$ for every $t \in \mathscr{T}$, $s \in \mathscr{S}$.

(ii) For $f, g \in \mathscr{E}(X)$, $h, i \in \mathscr{E}(Y)$, if $f \perp g$, then

$$\alpha(f + g, h) = \alpha(f, h) + \alpha(g, h)$$

and if $h \perp i$, then

$$\alpha(f, h + i) = \alpha(f, h) + \alpha(f, i)$$

We say that β or α is *algebraic* if for all $t_1, t_2 \in \mathscr{T}$, $s_1, s_2 \in \mathscr{S}$, $f \in \mathscr{E}(X)$, $g \in \mathscr{E}(Y)$ we have $\hat{\beta}(t_1, g) \approx \hat{\beta}(t_2, g)$, $\hat{\beta}(f, s_1) \approx \hat{\beta}(f, s_2)$ or $\alpha(t_1, g) \approx \alpha(t_2, g)$, $\alpha(f, s_1) \approx \alpha(f, s_2)$, respectively. The next results summarize some important properties of bimorphisms and effect bimorphisms.

Lemma 2.3. If β: $X \times Y \to Z$ is a bimorphism, then $\hat{\beta}$: $\mathscr{E}(X) \times \mathscr{E}(Y) \to \mathscr{E}(Z)$ is an effect bimorphism. Also, β is algebraic if and only if $\hat{\beta}$ is algebraic.

Theorem 2.4. Let (X, $\mathscr{T}$), $(Y, \mathscr{S})$, $(Z, \mathscr{U})$ be algebraic E-test spaces. If α: $\mathscr{E}(X) \times \mathscr{E}(Y) \to \mathscr{E}(Z)$ is an algebraic effect bimorphism, then $\tilde{\alpha}$: $\Pi(X) \times \Pi(Y) \to \Pi(Z)$ defined by

$$\tilde{\alpha}(\pi(f), \pi(g)) = \pi(\alpha(f, g))$$

is an effect algebra bimorphism.

Let $(X, \mathscr{T})$, $(Y, \mathscr{S})$, $(Z, \mathscr{U})$ be E-test spaces. If τ: $X \times Y \to Z$ is an (algebraic) bimorphism, we call $(Z, \mathscr{U}, \tau)$ an (*algebraic*) *tensor product* of $(X, \mathscr{T})$ and $(Y, \mathscr{S})$ if τ is surjective and for every (algebraic) E-test space $(V, \mathscr{W})$ and (algebraic) bimorphism β: $X \times Y \to V$, there exists a morphism ϕ: $Z \to V$ such that $\beta = \phi \circ \tau$.

Lemma 2.5. If an (algebraic) tensor product of (algebraic) E-test spaces exists, then it is unique to within an isomorphism.

Theorem 2.6. Let $(X, \mathscr{T})$, $(Y, \mathscr{S})$ be E-test spaces, If

$$\mathscr{T} \times \mathscr{S} = \{t \times s: t \in \mathscr{T}, s \in \mathscr{S}\}$$

then $(X \times Y, \mathscr{T} \times \mathscr{S}, \tau)$ is their tensor product, where τ is the identity mapping on $X \times Y$.

Theorem 2.7. The algebraic tensor product of algebraic E-test spaces $(X, \mathscr{T})$, $(Y, \mathscr{S})$ exists if and only if there exists an algebraic E-test space $(Z, \mathscr{U})$ and an algebraic bimorphism β: $X \times Y \to Z$.

Corollary 2.8. (a) If the algebraic tensor product of algebraic E-test spaces $(X, \mathscr{T})$, $(Y, \mathscr{S})$ exists, then the effect algebra tensor product of $\Pi(X)$ and $\Pi(Y)$ exists. (b) There exist algebraic E-test spaces that do not admit an algebraic tensor product.

The proof of Corollary 2.8(b) follows from the fact that there are effect algebras whose tensor products do not exist (Gudder and Greechie, 1996).

3. UNIVERSAL GROUPS

Let $(X, \mathcal{T})$ be an E-test space and let G be an additive abelian group. A mapping $\phi: \mathcal{E}(X) \to G$ is a *group-valued measure* if $f, g \in \mathcal{E}$ with $f \perp g$ implies $\phi(f + g) = \phi(f) + \phi(g)$. A *universal group* for $(X, \mathcal{T})$ is a pair (G, γ), where $\gamma: \mathcal{E} \to G$ is a group-valued measure that satisfies the following conditions:

(i) $\gamma(\mathcal{E})$ generates G.

(ii) If $\phi: \mathcal{E} \to H$ is a group-valued measure, then there exists a group homomorphism $\psi: G \to H$ such that $\phi = \psi \circ \gamma$.

Lemma 3.1. Universal groups are unique to within a group isomorphism.

Theorem 3.2. For an E-test space $(X, \mathcal{T})$, let G be the subgroup of $\mathbb{Z}^X$ generated by $\mathcal{E}(X)$ and let $\gamma: \mathcal{E}(X) \to G$ be the injection $\gamma(f) = f$. Then (G, γ) is the universal group for $(X, \mathcal{T})$.

Applying the two previous results, an E-test space $(X, \mathcal{T})$ has a unique (to within a group isomorphism) universal group (G, γ). We call (G, γ) of Theorem 3.2 *the* universal group for $(X, \mathcal{T})$. For $f, g \in G$, define $f \leq g$ if $f(x) \leq g(x)$ for all $x \in X$. Then $(G, \leq)$ becomes an abelian partially ordered group with generating positive cone $G^+ = \{f \in G: f \geq f_0\}$. Since

$$\{f \in G^+: f \leq t \text{ for some } t \in \mathcal{T}\}$$

generates G^+, $\mathcal{T}$ is a generating antichain in G. This shows that $(G, \mathcal{T})$ is a *test group* (Foulis *et al.*, n.d.).

Two test groups $(G_1, \mathcal{T}_1)$ and $(G_2, \mathcal{T}_2)$ are *isomorphic* if there exists an order and group isomorphism $\psi: G_1 \to G_2$ such that $\psi(\mathcal{T}_1) = \mathcal{T}_2$. Unlike effect algebras, for which two nonisomorphic effect algebras can have the same universal group, the universal group of an E-test space $(X, \mathcal{T})$ determines $(X, \mathcal{T})$ to within an isomorphism.

Theorem 3.3. Two E-test spaces are isomorphic if and only if their universal groups are isomorphic test groups.

Let (G, γ) be the universal group of $(X, \mathcal{T})$ and let H be the subgroup of G generated by

$$N = \{f - g: f, g \in \mathcal{E}, f \approx g\}$$

Let $\hat{G} = G/H$ and denote elements of $\hat{G}$ by $[g]$, $g \in G$. Assume $(X, \mathcal{T})$ is algebraic and define $\hat{\gamma}: \Pi(X) \to \hat{G}$ by $\hat{\gamma}(\pi(f)) = [f]$. Notice that $\hat{\gamma}$ is well defined because $g \approx f$ implies $[g] = [f]$.

Theorem 3.4. If $(X, \mathcal{T})$ is algebraic, then $(\hat{G}, \hat{\gamma})$ is the universal group of $\Pi(X)$.

As a corollary, we obtain the following fundamental result (Foulis and Bennett, 1994).

Corollary 3.5. An effect algebra admits a universal group.

4. COMPATIBILITY AND OBSERVABLES

For an E-test space $(X, \mathcal{T})$, if $t \in \mathcal{T}$, we define $\mathscr{E}_t = \{f \in \mathscr{E}: f \leq t\}$. We say that a set $\mathscr{S} \subseteq \mathscr{E}$ is *compatible* if $\mathscr{S} \subseteq \mathscr{E}_t$ for some $t \in \mathcal{T}$. If f, g are compatible, we write $f \leftrightarrow g$ and it is clear that $\leftrightarrow$ is a reflexive, symmetric relation. The next lemma summarizes some of the elementary properties of compatibility.

Lemma 4.1. (a) $f \leftrightarrow f_0$ (b) $f \leftrightarrow t \in \mathcal{T}$ if and only if $f \leq t$. (c) For $s, t \in \mathcal{T}$, $s \leftrightarrow t$ if and only if $s = t$. (d) If $f \leftrightarrow g$, $f_1 \leq f$, and $g_1 \leq g$, then $f_1 \leftrightarrow g_1$. (e) $\mathscr{S} \subseteq \mathscr{E}$ is compatible if and only if $\vee\{f: f \in \mathscr{S}\} \leq t$ for some $t \in \mathcal{T}$.

A finite number of $f_i \in \mathscr{E}$, $i = 1, \ldots, n$, are *orthogonal* if $\Sigma f_i \leq t$ for some $t \in \mathcal{T}$. An indexed family f_α, $\alpha \in A$, is *orthogonal* if every finite number of the f_α's are orthogonal. The following result gives a relationship between compatibility and orthogonality.

Theorem 4.2. (a) f_α, $\alpha \in A$, is orthogonal if and only if for every $x \in X$, $f_\alpha(x) = 0$ except for a finite number of $\alpha \in A$ and $\Sigma f_\alpha \leq t$ for some $t \in \mathcal{T}$. (b) If f_α, $\alpha \in A$, is orthogonal, then the set of different f_α's is compatible. (c) $\mathscr{S} \subseteq \mathscr{E}$ is compatible if and only if there exists an orthogonal family f_α, $\alpha \in A$, such that

$$f = \Sigma\{f_\alpha: \alpha \in B \subseteq A\}$$

for every $f \in \mathscr{S}$.

We denote the family of Borel subsets of $\mathbb{R}$ by $\mathscr{B}(\mathbb{R})$. An *observable* on an E-test space $(X, \mathcal{T})$ is a mapping $F: \mathscr{B}(\mathbb{R}) \to \mathscr{E}$ that satisfies the following conditions.

(i) $F(\mathbb{R}) \in \mathcal{T}$.

(ii) If $A_i \in \mathscr{B}(\mathbb{R})$, $i \in \mathbb{N}$, are mutually disjoint, then $F(\cup A_i) = \Sigma F(A_i)$, where the summation converges pointwise.

We now summarize some of the elementary properties of observables.

Lemma 4.3. Let F be an observable on $(X, \mathcal{T})$ with $F(\mathbb{R}) = t \in \mathcal{T}$. (a) If $A \subseteq B$, then $F(A) \leq F(B)$ and $F(B) = F(A) + F(B \setminus A)$. (b) $F(A) \in \mathscr{E}_t$, so $\{F(A): A \in \mathscr{B}(\mathbb{R})\}$ is compatible. (c) $F(\emptyset) = f_0$. (d) $F(A \cup B) = F(A)$

$+ F(B) - F(A \cap B)$. (e) If A_i are mutually disjoint, then $F(A_i)$ are orthogonal. (f) If A_i is a decreasing (increasing) sequence, then $\lim F(A_i) = F(\cap A_i)$ $(F(\cup A_i))$ pointwise.

Two observables F, G are compatible ($F \leftrightarrow G$) if $F(\mathbb{R}) = G(\mathbb{R})$. If F is an observable and u: $\mathbb{R} \to \mathbb{R}$ is a Borel function, define the observable $u(F)$: $\mathcal{B}(\mathbb{R}) \to \mathcal{E}$ by $u(F)(A) = F(u^{-1})(A))$. If $F(\{\lambda\}) \neq f_0$, then λ is an *eigenvalue* of F and $F(\{\lambda\})$ is the corresponding *eigeneffect.* We denote the set of eigenvalues of F by $\sigma_p(F)$ and we call $\sigma_p(F)$ the *point spectrum* of F. In general, $\sigma_p(F)$ need not be a Borel set.

Lemma 4.4. (a) $F \leftrightarrow G$ if and only if $F(A) \leftrightarrow F(B)$ for all $A, B \in \mathcal{B}(\mathbb{R})$. (b) If $F = u(H)$, $G = v(H)$, then $F \leftrightarrow G$, $F \leftrightarrow H$, $G \leftrightarrow H$.

Theorem 4.5. If F is an observable and u a Borel function, then $\sigma_p(u(F)) = u(\sigma_p(F))$.

We now consider various types of observables. An observable F is *atomic* if for every $A \in \mathcal{B}(\mathbb{R})$ we have

$$F(A) = \Sigma\{F(\{\lambda\}): \lambda \in A \cap \sigma_p(F)\}$$

We call F a *universal observable* for $t \in \mathcal{T}$ if $F(\mathbb{R}) = t$ and for any observable G such that $G(\mathbb{R}) = t$ there exists a Borel function u such that $G = u(F)$. We call F a *maximal observable* for $t \in \mathcal{T}$ if for every $f \in \mathcal{E}_t$ there exists an $A \in \mathcal{B}(\mathbb{R})$ such that $F(A) = f$.

Theorem 4.6. (a) Every observable is atomic. (b) Every universal observable is maximal.

The converse of Theorem 4.6(b) does not hold. For $t \in \mathcal{T}$ we define the *support* $S(t)$ of t by $S(t) = \{x \in X: t(x) \neq 0\}$, and denoting the cardinality of a set A by $|A|$, we define

$$\text{card}(t) = |\Sigma\{t(x): x \in S(t)\}|$$

Notice that $|S(t)| \leq \text{card}(t)$ and if $|S(t)|$ is infinite, then $|S(t)| = \text{card}(t)$.

Theorem 4.7. (a) $t \in \mathcal{T}$ admits a maximal observable if and only if $\text{card}(t) \leq |\mathbb{R}|$. (b) If F is a universal observable for t, then $|\sigma_p(F)| = \text{card}(t)$. (c) If $t \in \mathcal{T}$ admits a universal observable, then $\text{card}(t) \leq |\mathbb{R}|$.

It is unknown whether the converse of Theorem 4.7(c) holds. There are many important examples in which card $(t) \leq |\mathbb{N}|$. For instance, consider an algebraic E-test space whose logic is the standard effect algebra on a separable Hilbert space. We now show that if card $(t) \leq |\mathbb{N}|$, then strong results can be obtained.

Theorem 4.8. If card(t) $\leq |\mathbb{N}|$, then t admits a universal observable F and F is unique to within a Borel isomorphism.

Corollary 4.9. It card (t) $\leq |\mathbb{N}|$ and F, G are observables for t, then there exists an observable H and Borel functions u, v such that $F = u(H)$, $G = v(H)$.

If P is an effect algebra, an *effect algebra observable* on P is a morphism F: $\mathcal{B}(\mathbb{R}) \to P$. That is, $F(\mathbb{R}) = 1$ and $A, B \in \mathcal{B}(\mathbb{R})$ with $A \cap B = \emptyset$ imply that $F(A) \perp F(B)$ and $F(A \cup B) = F(A) \oplus F(B)$.

Theorem 4.10. If (X, $\mathcal{T}$) is an algebraic E-test space and F: $\mathcal{B}(\mathbb{R}) \to \mathcal{E}$ is an observable, then $\hat{F}$: $\mathcal{B}(\mathbb{R}) \to \Pi(X)$ given by $\hat{F}(A) = \pi(F(A))$ is an effect algebra observable.

An effect f is *sharp* in $t \in \mathcal{T}$ if $f \leq t$ and $f(x) = t(x)$ for all $x \in S(f)$. For $t \in \mathcal{T}$, form the Hilbert space $\mathcal{H}_t$ consisting of the functions ψ: $S(t) \to \mathbb{C}$ such that $\Sigma|\psi(x)|^2 < \infty$. For $f \in \mathcal{E}_t$, define the bounded linear operator $\hat{f}$ on $\mathcal{H}_t$ by $(\hat{f}\psi)(x) = [f(x)/t(x)]\,\psi(x)$. It is clear that $\hat{\mathcal{E}}_t = \{\hat{f}: f \in \mathcal{E}\}$ is a subset of the standard Hilbert space effect algebra on $\mathcal{H}_t$. The next result shows that in this case, observables correspond to the usual positive operator-valued (POV) measures of Hilbert space quantum mechanics.

Theorem 4.11. (i) If F is an observable for t, then $\hat{F}$ is a POV measure on $\mathcal{H}_t$. (ii) $f \in \mathcal{E}_t$ is sharp if and only if $\hat{f}$ is a projection.

We close this article with a discussion of the spectrum of an observable F. Physically, the spectrum of F is the set of values that F can attain arbitrarily closely.

Theorem 4.12. If F is an observable, then there exists a largest open set $A \subset \mathbb{R}$ such that $F(A) = f_0$.

The complement A^c of the above set A in Theorem 4.12 is the *spectrum* of F and is denoted $\sigma(F)$. Clearly, $\sigma(F)$ is the smallest closed set $B \subseteq \mathbb{R}$ such that $F(B) = t$ for some $t \in \mathcal{T}$.

Lemma 4.13. A real number λ is an element of $\sigma(F)$ if and only if for every open set A such that $\lambda \in A$, we have $F(A) \neq f_0$.

There are simple examples which show that $\sigma_p(F) \neq \sigma(F)$ in general. Nevertheless, there is a close relationship between $\sigma_p(F)$ and $\sigma(F)$. We denote the closure of a set $A \subseteq \mathbb{R}$ by $\bar{A}$.

Our last result is called a spectral mapping theorem. This theorem gives a relationship between $\sigma(u(F))$ and $u(\sigma(F))$.

Theorem 4.15. Let F be an observable and $u: \mathbb{R} \to \mathbb{R}$ be a Borel function. (a) $\sigma(u(F)) \subseteq u(\sigma(F))$. (b) If u is continuous, then $\sigma(u(F)) = u(\sigma(F))$. (c) If $\sigma(F)$ is bounded and u is continuous on $\sigma(F)$, then $\sigma(u(F)) = u(\sigma(F))$.

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